Reiman stieltjes condition

In mathematics, the **Riemann–Stieltjes integral** is a generalization of the Riemann integral, named after Bernhard Riemann and Thomas Joannes Stieltjes. The definition of this integral was first published in 1894 by Stieltjes.[1] It serves as an instructive and useful precursor of the Lebesgue integral, and an invaluable tool in unifying equivalent forms of statistical theorems that apply to discrete and continuous probability.

Defination :-

if f€R (α) on [a,b] iff for every ε> 0, there exists a partition P of [a,b] such that U(P,f,α)-L(P,f,α)< ε

Proof:-

Let f€R (α) on [a,b], then

$$\underline{\int_a^b} f\,d\alpha = \overline{\int_a^b} f\,d\alpha = \int_a^b f\,d\alpha \qquad \text{........(1)}$$

Let €>0 be given number and since the upper and lower integral are the sup and inf respectively of the upper and lower sum , therefore there exists partition P_1 and P_2 such that

$$U(P_1, f, \alpha) < \overline{\int_a^b} f\, d\alpha + \frac{\in}{2}$$

$$\text{or } U(P_1, f, \alpha) < \int_a^b f\, d\alpha + \frac{\in}{2} \quad \text{(By 1)}$$

$$\text{Also } U(P_2, f, \alpha) > \underline{\int_a^b} f\, d\alpha + \frac{\in}{2}$$

By (1), we have

$$U(P_2, f, \alpha) > \int_a^b f\, d\alpha - \frac{\epsilon}{2}$$

Now if $P = P_1 \cup P_2$ is the common Refinement of P_1 and P_2 , then

$$U(P, f, \alpha) \leq U(P_1, f, \alpha) < \int_a^b f\, d\alpha + \frac{\epsilon}{2} < L(P_2, f, \alpha) + \frac{\epsilon}{2} \leq L(P, f, \alpha) + \frac{\epsilon}{2}$$

Hence $U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$

Conversely:-

Let U(P,f,α)-L(P,f,α)< ε.

Now we show that f€R (α) over [a,b]

By defination

$$L(P,f,\alpha)\leq \underline{\int_a^b} f\,d\alpha \leq \overline{\int_a^b} f\,d\alpha \leq U(P,f,\alpha)$$

$$\therefore \quad 0\leq \overline{\int_a^b} f\,d\alpha - \underline{\int_a^b} f\,d\alpha \leq U(P,f,\alpha)-L(P,f,\alpha) < \in$$

$$\Rightarrow \int_{a}^{\overline{b}} f\,d\alpha - \int_{\underline{a}}^{b} f\,d\alpha < \epsilon$$

Since € is arbitrary , we have

$$\int_{a}^{\overline{b}} f\,d\alpha = \int_{\underline{a}}^{b} f\,d\alpha$$

$\Rightarrow$ f€ R(α) over [a,b]

Hence the proof

Prove that upper Reiman sum is grater than lower Reiman sum

$$\underline{\int_a^b} f\, d\alpha \leq \overline{\int_a^b} f\, d\alpha$$

Proof : Let P_1 and P_2 be the partition of [a,b]

Let P^* be the common refinement of P_1 and P_2, then by the theorem " If P^* is the refinement of P , then

(i) $L(P_1,f,a) \leqslant L(P^*,f,a)$ (ii) $U(P^*,f,a) \leqslant U(P,f,a)$

we have $L(P_1,f,a) \leqslant L(P^*,f,a) \leqslant U(P^*,f,a) \leqslant U(P_2,f,a)$

$\Rightarrow L(P_1,f,a) \leqslant U(P_2,f,a)$ ......(1)

Keeping P_2 fixed and lub over P_1, we have from (1)

l.u.b $L(P_1,f,\alpha) \leqslant U(P_2,f,\alpha)$

$$\Rightarrow \underline{\int_a^b} f\,dx \le U(P_2, f, \alpha) \qquad \text{.....(2)}$$

Now taking g.l.b of all partition P_2, we get from (2)

$$\overline{\int_a^b} f\,d\alpha \le \text{g.l.b.}\,U(P_2, f, \alpha)$$

$$\Rightarrow \quad \underline{\int_a^b} f \, d\alpha \leq \overline{\int_a^b} f \, d\alpha$$

Hence the proof