Mathematics Presentation

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Cauchy's First Theorem on Limits

❖ Statement :

If $a_n \to l$, then $x_\eta = \frac{a_1+a_2+a_3+\cdots+a_n}{n} \to l$

Proof.

If $t_n = a_n - l$ i.e., $a_n = t_n + l$

Let $x = \frac{a_1+a_2+a_3+\cdots a_n}{} = (t_1 + \text{l}) + (t_2+\text{l}) + (t_3 + l) + \cdots + (\text{t} + \text{l})$

$$= \frac{nl+(t_1 + t_2+ t_3+\cdots+ t_n)}{n}$$

Therefore $x = l + \dfrac{t_1 + t_3 + \cdots + t_n}{n}$ ....(1)

Now $a_n \to l$, $t_n \to 0$

Therefore given $\in > 0$, however small , there exists a positive integer m such that

$|tn| < \frac{\text{g}}{2}$ $for\ all$ $n \geq m$ … … … … (2)

Again as $t_n \to 0,$ $Therefore$ $\{t_n\}$ $is\ convergent$

$$\Rightarrow \quad \{t_n\}\ is\ bounded$$

Therefore there exists a real number k such that

$$|tn| < k \ \forall\ n \qquad \ldots\ldots\ldots.(3)$$

Now $\left|\frac{t_1+t_2+t_3+\cdots+t_n}{n}\right| = \left|\frac{t_1+t_2+t_3+t_m+t_{m+1}+\cdots+t_n}{}\right|$

$$= \left|\frac{t_1+t_2+\cdots+t_m}{n} + \frac{t_{m+1}+\cdots+t_n}{n}\right|$$

$$\leq \frac{|t_1 + t_2 + \cdots + t_m|}{n} + \frac{|t_{m+1} + \cdots + t_n|}{n}$$

$$\leq \frac{|t_1| + |t_2| + \cdots + |t_m|}{n} + \frac{|t_{m+1}| + \ldots + |t_n|}{n}$$

$$< \frac{mk}{n} + \frac{n-m}{n} \cdot \frac{g}{2}$$ [Because of (2) and (3)]

$$< \frac{mk}{n} + \frac{g}{2}$$

Therefore $\left|\frac{t_1+t_2+t_3+\cdots+t_n}{n}\right| < \frac{mk}{n} + \frac{g}{2}$ ……….(4)

Now $\frac{mk}{n} \to 0$ as $n \to \infty$

Therefore there exists a positive integer p such that $\frac{mk}{n} < \frac{g}{2}$ $for \;\; n \geq p$ ………….(5)

Let q = Max.(m , p)

$$\Rightarrow \left|\frac{t_1+t_2+t_3+\cdots+t_n}{n} - 0\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} \quad for \quad n \geq q$$ [because of (4) and (5)]

$$\Rightarrow \left|\frac{t_1+t_2+t_3+\cdots+t_n}{n} - 0\right| < \epsilon \quad for \quad n \geq q$$

from (1), $\lim_{n\to\infty} x_n = l$

$$\Rightarrow \quad \lim_{n\to\infty} \frac{a_1+a_2+a_3+\cdots+a_n}{n} = l$$

Cauchy’s Second Theorem on Limits

❖ Statement :

If $\{a_n\}$ is a sequence of positive terms and $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ exists whether finite or infinite, then $\lim_{n\to\infty} (a_n)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$

Proof. Two cases arise:

Case 1. $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = l$, where l is finite

Therefore given $\epsilon > 0$, however small, $\exists\ m \in N$ such that

$$\left|\frac{a_{n+1}}{a_n} - l\right| < \frac{\epsilon}{2}\ \forall\ n \geq m$$

$$\Rightarrow \quad l - \frac{g}{2} < \frac{a_{n+1}}{a_n} < l + \frac{g}{2} \quad \forall\ n \geq m$$

Putting n=m, m+1, m+2,.......,n-1, we get,

$$l-\frac{\epsilon}{2}<\frac{a_{m+1}}{a_m}<l+\frac{\epsilon}{2}$$

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$$l-\frac{\epsilon}{2}<\frac{a_{m+2}}{a_{m+1}}<l+\frac{\epsilon}{2}$$

$$l-\frac{\epsilon}{2}<\frac{a_{\mathrm{m}\ +3}}{a_{m+2}}<l+\frac{\epsilon}{2}$$

…. . …. . . …. . . …. . . …. . . …. . . …. . …. . . …. .

$$l-\frac{\epsilon}{2}<\frac{a_n}{a_n-1}<l+\frac{\epsilon}{2}$$

Multiplying these inequations, we get,

$$\left(l-\frac{\epsilon}{2}\right)^{n-m} < \frac{a_n}{a_m} < \left(l+\frac{\epsilon}{2}\right)^{n+m}$$

$$\Rightarrow \quad a_m\left(l-\frac{\epsilon}{2}\right)^{n-m} < a_n < a_m\left(l+\frac{\epsilon}{2}\right)^{n+m}$$

$$\Rightarrow \quad (a_m)^{\frac{1}{n}}\left(l-\frac{\epsilon}{2}\right)^{\frac{n-m}{n}} < (a_n)^{\frac{1}{n}} < (a_m)^{\frac{1}{n}}\left(l+\frac{\epsilon}{2}\right)^{\frac{n+m}{n}}$$

$$\Rightarrow \quad (a_m)^{\frac{1}{n}}\left(l+\frac{g}{2}\right)^{1-mT_n} < (a_n)^{\frac{1}{n}} < (a_m)^{\frac{1}{n}}\left(l+\frac{g}{2}\right)^{1+\frac{m}{n}} \qquad \ldots\ldots\ldots\ldots (1)$$

Let $n \to \infty$

Now $(a_m)^{\frac{1}{}}\left(l-\frac{g}{2}\right)^{1-\frac{m}{}} \to l-\frac{g}{2}$

[Because $(a_m)^{\frac{1}{n}} \to 1 \text{ as } n \to \infty \text{ for } a > 0$]

and $(a_m)^{\frac{1}{}}\left(l+\frac{g}{2}\right) \to l+\frac{g}{2}$

Therefore given $\epsilon > 0, \exists\, m_1, m_2 \epsilon$ N such that

$$\left|(a_m)^{\frac{1}{}}\left(l-\frac{\epsilon}{2}\right)^{1-\frac{m}{}} - \left(l-\frac{\epsilon}{2}\right)\right| < \frac{\epsilon}{2}\ \forall\, n \geq m_1$$

and $$\left|(a_m)^{\frac{1}{}}\left(l+\frac{g}{2}\right)^{1+\frac{m}{}} - \left(l+\frac{g}{2}\right)\right| < \frac{g}{2}\ \forall\, n \geq m_2$$

Therefore $\left(l-\frac{g}{2}\right)-\frac{g}{2} < (a_m)^{\frac{1}{}}\left(l-\frac{g}{2}\right)^{1-\frac{m}{}} < \left(l-\frac{g}{2}\right)+\frac{g}{2}$

i.e., $l - \in < (a_m)^{\frac{1}{}} \left(l - \frac{g}{2}\right)^{1-\frac{m}{}} < l \;\forall\; n \geq m_1$ ……………. (2)

and $\left(l + \frac{g}{2}\right) - \frac{g}{2} < (a_m)^{\frac{1}{}} \left(l - \frac{g}{2}\right)^{1-\frac{m}{}} < \left(l + \frac{g}{2}\right) - \frac{g}{2}$

i.e., $l < (a_m)^{\frac{1}{}} \left(l + \frac{g}{2}\right)^{1-\frac{m}{}} < l + \epsilon \;\forall\; n \geq m_2$ ………. (3)

Let p = Max.(m, $m_{1,}m_2$)

Therefore from (1), (2), (3), we get

$$l - \frac{\epsilon}{2} < (a_n)^{\frac{1}{n}} < l + \in \forall n \geq p$$

or $\left|(a_n)^{\frac{1}{n}} - l\right| < \epsilon \; \forall \; n \geq p$

Therefore $\lim_{n\to\infty} (a_n)^{\frac{1}{n}} = l \quad or \quad \lim_{n\to\infty} (a_n)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{a_{n+1}}{a}$

Case 2. $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = +\infty$

Therefore $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = 0 \Rightarrow \lim_{n\to\infty} \frac{\frac{1}{a_{n+1}}}{\frac{1}{a_n}} = 0$ (finite)

Therefore $\lim_{n\to\infty} \left(\frac{1}{a}\right)^{\frac{1}{n}} = 0 \quad \Rightarrow \quad \lim_{n\to\infty} (a_n)^{\frac{1}{n}} = +\infty$