Math presentation

Name .Manpreet

Roll no . 5208

Topic. Cauchys theorem

Cauchys first Theorem On Limits

If $a_n \to l$ then $x_n = \dfrac{a_1+a_2+a_3+....+a_n}{n} \to l$

Proof : Let $t_n = a_n - l$ i.e,. t_n+l

Now $x_n = \dfrac{a_1+a_2+a_3+...+a_n}{n} =$

$$\frac{(t_1+l)+(t_2+l)+(t_3+l)+.....+(t_n+n)}{n}$$

$$= \frac{nl+(t_1+t_2+t_3+\ldots+t_n)}{n}$$

$$\therefore X_n = \frac{l+t_1+t_2+t_3+\ldots+t_n}{n} \qquad \ldots 1$$

Now $a_n \to l \qquad t_n \to 0$

$\therefore$ given $E > 0$, however small, there exist a positive integers m such that

$$|t_n| < \frac{E}{2}$$

Again as $t_n \to 0$ $\therefore \{t_n\}$ is convergent

$\{t_n\}$ is bounded

$\therefore$ There exist a real number k such that

$$|t_n| < k$$

Now

$$\frac{|t_1+t_2+t_3+\ldots+t_n|}{n} = \frac{|t_1+t_2+t_3+t_m+t_{m+1}+\ldots+t_n|}{n}$$

$$= \left|\frac{t_1+t_2+\ldots+t_m}{n} + \frac{t_{m+1}+\ldots+t_n}{n}\right|$$

$$\leq \frac{|t_1+t_2+\ldots+t_m|}{n} + \frac{|t_{m+1}+\ldots+t_n|}{n}$$

$$< \frac{mk}{n} + \frac{n-m}{n} . \frac{E}{2}$$

$$< \frac{mk}{n} + \frac{E}{2}$$

$$:. \left| \frac{t_1+t_2+t_3+.....+t_n}{n} \right| < \frac{mk}{n} + \frac{E}{2}$$

Now $\frac{mk}{n}$ 0 as n infinity

:. There exist a positive integers p such that $\frac{mk}{n}$,< $\frac{E}{2}$ for n $\geq$ p

Let q = Max . (m,p)

$$\therefore \left| \frac{t_1+t_2+t_3+\dots+t_n}{n} \right| \;,\; \frac{E}{2}+\frac{E}{2} \text{ for } n > q$$

$$\therefore \left| \frac{t_1+t_2+t_3+\dots+t_n - 0}{n} \right| < \frac{E}{2} \text{ for } n > q$$

$\therefore$ From 1 lt $x_n = l$

$= Lt \; \underline{a_1+a_2+a_3+\dots+a_n} \; = l$

note1 : sequence $\{ x_n \}$ is called the sequence of means of the sequence $\{ a_n \}$

Note2: the converse of the above result is not true .

Cauchys second theorem on limits

If {an} is sequence of positive terms and lt $\frac{an+1}{an}$

n → ∞

Exist whether finite or in finite , then $Lt \; an^{\frac{1}{n}}$ = Lt $\frac{an+1}{an}$

n → ∞

Proof : two cases arese

Case 1: lt $\frac{an+1}{an} = l$, where l is finite given $E < 0$, however small

$$\left| \frac{a_{n+1} - l}{a_n} \right| < \frac{E}{2}$$

$$\Rightarrow \quad L - \frac{E}{2} < \frac{a_{n+1}}{a_n} < l + \frac{E}{2}$$

Putting n = m , m+1 , m+2 ......, n- 1 , we get

$$L - \frac{E}{2} < \frac{a_{m+1}}{a_m} < l < \frac{E}{2}$$

$$l - \frac{E}{2} < \frac{a_{m+2}}{a_{m+1}} < l < \frac{E}{2}$$

$$l - \frac{E}{2} < \frac{am+3}{am+3} < l + \frac{E}{2}$$

….. …….. ……. …….

$$l - \frac{E}{2} < \frac{an}{an-1} < +\frac{E}{2}$$

Multiplying in these equation , we get

$$\left(1-\frac{E}{2}\right)^{n-n} < \frac{an}{am} < \left(1+\frac{\varepsilon}{2}\right)^{n-m}$$

$$am\left(1-\frac{\varepsilon}{2}\right)^{n-m} < an < am\left(1+\frac{\varepsilon}{2}\right)^{n-m}$$

$$\Rightarrow (am)^{\frac{1}{n}}\left(l-\frac{E}{2}\right)^{n}\frac{-m}{n} < (an)^{\frac{1}{n}} < (am)^{\frac{1}{n}}\left(L+\frac{\varepsilon}{2}\right)^{n}\frac{-m}{n}$$

$$\Rightarrow (am)^{\frac{1}{n}}\left(1-\frac{\varepsilon}{2}\right)^{L-\frac{m}{n}} < (an)^{\frac{1}{n}} < (am)^{\frac{1}{n}}\left(L+\frac{\varepsilon}{2}\right)\frac{l-m}{n}$$

Let $n \to \infty$

Now $(am)^{\frac{1}{n}}\left(L-\frac{\varepsilon}{2}\right)^{\frac{L-m}{n}} \to L-\frac{\varepsilon}{2}$

And $(am)^{\frac{1}{n}}\left(l+\frac{\Sigma}{2}\right)^{L+\frac{n_7}{n}} \longrightarrow L+\frac{\varepsilon}{2}$

∴ Given E > 0 , m1 ,m2 ε N s.t.

$$\left|(am)^{\frac{1}{n}} \cdot \left(l-\frac{\varepsilon}{2}\right)^{L}\frac{-m}{m}-\left(L-\frac{\varepsilon}{2}\right)\right| \quad < \frac{\varepsilon}{2}\forall n \geq m_1$$

and $\lim^{\frac{1}{n}}\left(l+\frac{\varepsilon}{2}\right)^{L+\frac{m}{n}}-\left(L+\frac{\varepsilon}{2}\right)<\frac{\varepsilon}{2} \quad \forall n \geq m_2$

$$\therefore \left(L-\frac{\varepsilon}{2}\right)-\frac{\varepsilon}{2}<(am)^{\frac{1}{n}}\left(L-\frac{\varepsilon}{2}\right)<^{L-\frac{m}{n}}\left(L-\frac{\varepsilon}{2}\right)+\frac{\varepsilon}{2}$$

i.e. L-e < $(am)^{\frac{1}{n}}\left(L-\frac{\varepsilon}{2}\right)^{L-\frac{m}{n}}<l\forall n \geq m_1$

And $\left(-\frac{\varepsilon}{2}\right)-\frac{\varepsilon}{2}<(am)^{\frac{1}{m}}\left(L-\frac{E}{2}\right)^{L-\frac{m}{2}}<\left(L+\frac{\varepsilon}{2}\right)-\frac{\varepsilon}{2}$

i.e, l- $(am)^{\frac{1}{n}}\left(1+\frac{\Sigma}{2}\right)^{L-\frac{m}{n}}<1+$ e$Vn \geq m_2$

Let p = max . (m, m1, m2)

:. From (1) (2) (3) , we get

$$l - \varepsilon < (an)^{\frac{1}{n}} < L + \varepsilon \forall n \geq P$$

Or $\left|\{an\}^{\frac{1}{n}} - l\right| < \varepsilon \forall n \geq p$

$$:. Lt(an)^{\frac{1}{n}} = L$$

$$n \to \infty$$

Another form : if an > 0 and $\frac{an+1}{a\eta} \to L$ then prove that

$$n\sqrt{an} \to L$$

Case 2 : $Lt\frac{an+1}{a^n} = +\infty$

$$\eta \to \infty$$

$$\therefore \lim_{n \to \infty} \frac{an}{an+1} = 0 \Rightarrow \lim_{n \to \infty} \frac{\frac{1}{an+1}}{-I_{an}} = 0$$

$$\therefore \lim_{n \to \infty} \left(\frac{1}{a}n\right)^{\frac{1}{n}} = 0 \Rightarrow \lim_{n \to \infty} Lt(an)\frac{1}{n} = +\infty$$

Note : The converse of the above theorem is not true .