MATHEMATICS PRESENTATION VECTOR INTEGRATION

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SECTION-I
INTEGRATION OF VECTORS

Art-1. Indefinite Integral

We know that integration is the reverse process of differentiation.

Let $\vec{f}(t)$ and $\vec{F}(t)$ be two vector functions such that

$\frac{d}{dt}\left\{\vec{F}(t)\right\} = \vec{f}(t)$. Then $\vec{F}(t)$ is called the indefinite integral of $\vec{f}(t)$ w.r.t. t and we write it as $\int \vec{f}(t)\, dt = \vec{F}(t) + \vec{c}$

Art-2. If vector function $\vec{F}$ is indefinite integral of $\vec{f}$ w.r.t. t, then prove that $\vec{F} + \vec{c}$ is also indefinite integral of $\vec{f}$, where $\vec{c}$ is a constant vector.

Proof : Here $\int \vec{f}\, dt = \vec{F}$

$$\therefore \quad \frac{d\vec{F}}{dt} = \vec{f} \qquad \ldots(1)$$

Now, $\frac{d}{dt}(\vec{F} + \vec{c}) = \frac{d\vec{F}}{dt}$ $\left[\vec{c} \text{ being a constant vector, } \frac{d\vec{c}}{dt} = 0\right]$

or $\frac{d}{dt}(\vec{F} + \vec{c}) = \vec{f}$ $[\because \text{ of } (1)]$

$$\therefore \quad \int \vec{f}\, dt = \vec{F} + \vec{c}$$

Note : 1. The constant of integration c is scalar if integrand is scalar and vector if integrand is vector.

Note : 2. If $\vec{f} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$, where f_1, f_2, f_3 are scalar functions of some variable t say, then

$$\int \vec{f}\, dt = \hat{i}\int f_1\, dt + \hat{j}\int f_2\, dt + \hat{k}\int f_3\, dt.$$

Example 4. Evaluate $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = xy\hat{i} + yz\hat{j} + zx\hat{k}$ and curve C is $\vec{r} = t\hat{i} + t^2\hat{j} + t^3\hat{k}$, t varies from -1 to 1.

Sol. Here $\vec{F} = xy\hat{i} + yz\hat{j} + zx\hat{k}$, $\vec{r} = t\hat{i} + t^2\hat{j} + t^3\hat{k}, t$

$$\therefore \quad \frac{d\vec{r}}{dt} = \hat{i} + 2t\hat{j} + 3t^2\hat{k}$$

Also $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \Rightarrow x = t, y = t^2, z = t^3$

$$\therefore \quad \vec{F} = t^3\hat{i} + t^5\hat{j} + t^4\hat{k}$$

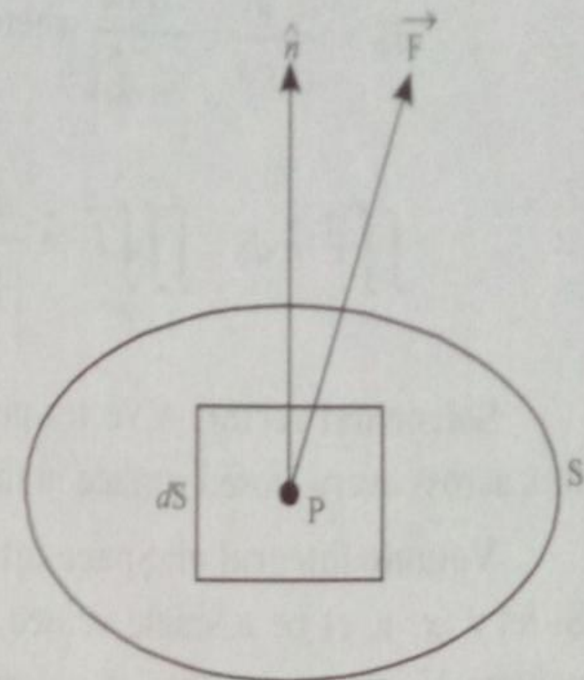
$$\therefore \quad \int_C \vec{F}\cdot d\vec{r} = \int_{-1}^{1} \vec{F}.\frac{d\vec{r}}{dt}\,dt = \int_{-1}^{1}(t^3 + 2t^6 + 3t^6)\,dt = \int_{-1}^{1}(t^3 + 5t^6)\,dt$$

$$= \left[\frac{t^4}{4}\right]_{-1}^{1} + \left[\frac{5t^7}{7}\right]_{-1}^{1} = \left(\frac{1}{4} - \frac{1}{4}\right) + \left(\frac{5}{7} + \frac{5}{7}\right) = 0 + \frac{10}{7} = \frac{10}{7}$$

(1)

Flux across a surface

Let S be a piecewise smooth surface and $\vec{F}(x, y, z)$ be a vector function of position defined and continuos over S. Let P be any point on the surface S and $\hat{n}$ be the unit vector at P in the direction of outward drawn normal to the surface S at P. Then $\vec{F} \cdot \hat{n}$ is the normal component of $\vec{F}$ at P. The integral of $\vec{F} . \hat{n}$ over S *i.e.*, $\iint_S \vec{F}.\hat{n}\, dS$ is called the flux of $\vec{F}$ over S.



Let $d\vec{S}$ be a vector (called vector area) of magnitude dS and direction that of $\hat{n}$. Then $d\vec{S} = \hat{n}\, dS$

$$\therefore \quad \iint_S \vec{F} \cdot \hat{n}\, dS = \iint_S \vec{F} \cdot d\vec{S}$$

Art-3. Definite Integral

If $\frac{d\vec{F}}{dt} = \vec{f}$ for all values of t in the interval $[a, b]$ then the definite integral between a and b is denoted as $\int_a^b \vec{f}\, dt$ and is defined as $\int_a^b \vec{f}\, dt = \left[\vec{F}\right]_a^b = \vec{F}(b) - \vec{F}(a)$.

Note : We give below some results which can be proved easily :

1. $\int \left(\frac{d\vec{f}}{dt} \cdot \vec{g} + \vec{f} \cdot \frac{d\vec{g}}{dt}\right) dt = \vec{f} \cdot \vec{g} + c$

2. $\int \left(\frac{d\vec{f}}{dt} \times \vec{g} + \vec{f} \times \frac{d\vec{g}}{dt}\right) dt = \vec{f} \times \vec{g} + c$

3. $\int \left(2\vec{r} . \frac{d\vec{r}}{dt}\right) dt = \vec{r}^{\,2} + c$

4. $\int \left(2\frac{d\vec{r}}{dt} . \frac{d^2\vec{r}}{dt^2}\right) dt = \left(\frac{d\vec{r}}{dt}\right)^2 + c$

5. $\int \left(\vec{r} \times \frac{d^2\vec{r}}{dt^2}\right) dt = \vec{r} \times \frac{d\vec{r}}{dt} + \vec{c}$

6. $\int \left(\vec{a} \times \frac{d\vec{r}}{dt}\right) dt = \vec{a} \times \vec{r} + \vec{c}$ where $\vec{a}$ is a constant vector.

7. $\int c\vec{r}\, dt = c\int \vec{r}\, dt$.

EXAMPLES

Art-6. Work Done and Circulation

Work done by a force : Let $\vec{F} = F_1\hat{i} + F_2\hat{j}\ F_3\hat{k}$ be a force acting at P with position vector $x\hat{i} + y\hat{j} + z\hat{k}$.

Then the work done by the force $\vec{F}$ in displacing a unit particle from A to B is defined as line integral from A to B

$$\therefore \quad \text{Work done} = \int_A^B \vec{F}\cdot\hat{t}\,ds = \int_A^B \vec{F}\cdot d\vec{r} = \int_A^B (F_1\,dx + F_2 dy + F_3 dz)$$

Conservative Field. A force $\vec{F}$ is said to be conservative if the work done by it in moving its point of application from a point A to B depends only on the points A and B and not upon the path joining A and B.

Circulation. If C is a closed curve, then the tangential line integral of $\vec{F}$ along C is called the circulation of $\vec{F}$ along C.

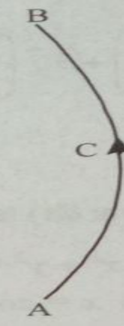
$$\therefore \quad \text{circulation of } \vec{F} \text{ along } C = \int_C \vec{F}\cdot d\vec{r} = \int_C (F_1 dx + F_2\,dy + F_3 dz)$$

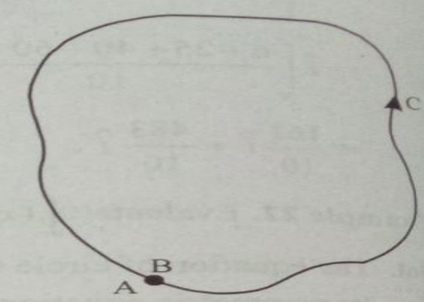
SECTION-II

TANGENTIAL LINE INTEGRAL

Art-4. Some Preliminary Concepts

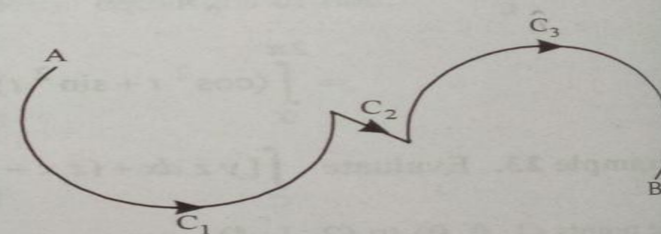
Oriented Curve : Let C be any curve in space. Let us orient C by taking one of two directions along C as the positive direction ; the opposite direction along C is then called the negative direction. Suppose A is the initial point and B the terminal point of C under the chosen orientation. If A and B coincide, then the curve C is called a closed curve.





Smooth Curve : A curve $\vec{r} = \vec{f}(t)$ is called smooth if $\vec{f}(t)$ is continuously differentiable. In other words, a curve is said to be smooth if it possesses a unique tangent at each of its points.

A curve C is said to be **piecewise smooth** if it is composed of a finite number of smooth curves. The curve C shown in the above figure is piecewise smooth as it is composed of three smooth curves C_1, C_2 and C_3. The circle is smooth closed curve while the curve consisting of the four sides of a rectangle is a piecewise smooth closed curve.

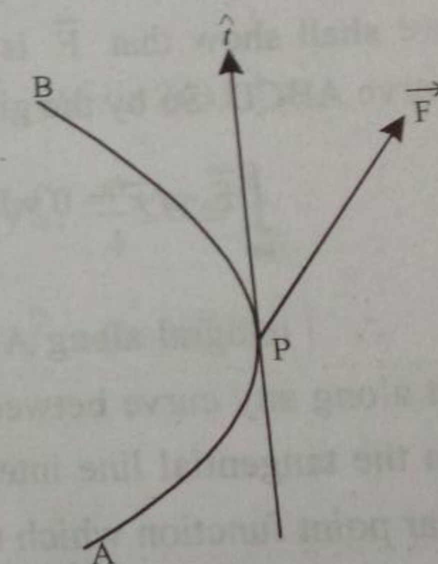


Smooth Surface : A surface S, which has a unique normal at each of its points and the direction of this normal depends continuously on the points of S, is called a smooth surface.

If a surface S is not smooth but can be subdivided into a finite number of smooth surfaces, then it is called a **piecewise smooth surface**. The surface of a sphere is smooth while the surface of a cube is piecewise smooth.

Tangential Line Integral : The tangential line integral of a vector function $\vec{F}$ along a curve C from A to B is the definite integral of the scalar resolute of $\vec{F}$ in the direction of the tangent to the curve measured from a fixed point in the sense A to B, and the limits of integration being the values of *s* corresponding to the points A and B.

If $\hat{t}$ is the unit tangent at the point P and $\vec{F}$ is the value of the function here, then tangential line integral



$$= \int_A^B \vec{F}\cdot\hat{t}\,ds = \int_A^B \vec{F}.\frac{d\vec{r}}{ds}\,ds = \int_A^B \vec{F}\cdot d\vec{r}$$

$$= \int_A^B (F_1\hat{i} + F_2\hat{j} + F_3\hat{k}).(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$$

$$= \int_A^B (F_1\,dx + F_2\,dy + F_3\,dz)$$

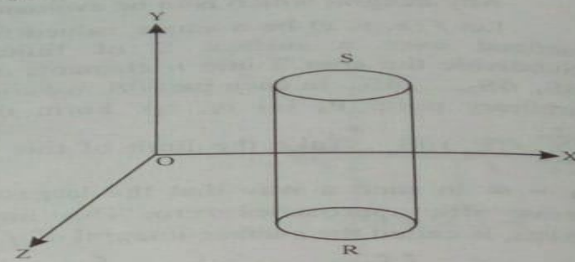
... integral of a vector function $\vec{F}$ vanishes for every

$$\therefore \quad \iint_S \vec{F}\cdot\hat{n}\,dS = \iint_S (F_1\cos\alpha + F_2\cos\beta + F_3\cos\gamma)\,dS$$

$$= \iint (F_1\,dy\,dz + F_2\,dz\,dx + F_3\,dx\,dy)$$

where $dy\ dz$, $dz\ dx$, $dx\ dy$ are the orthogonal projection of S on the co-ordinate planes.

Note. For evaluating surface integrals, it is easy to express them as double integrals taken over the orthogonal projection of the surface S on one of the co-ordinate planes. But this is possible only if axis line perpendicular to the co-ordinate plane chosen meets the surface in not more than one point.



Suppose the surfaces S is such that any line perpendicular to the xy-plane meets S in not more than one point. Let R be the orthogonal projection of S on the xy-plane. If γ is the acute angle which $\hat{n}$ at P to the surface S makes with z-axis, then

$$\cos\gamma\ dS = dx\ dy$$

where dS is the small element of area of surface S at P.

$$\therefore \quad dS = \frac{dx\,dy}{\cos\gamma} = \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$$ where $\hat{k}$ is the vector along z-axis.

$$\therefore \quad \iint_S \vec{F}\cdot\hat{n}\,dS = \iiint_R \vec{F}\cdot\hat{n}\,\frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$$

Solenoidal vector. A vector point function is said to be solenoidal in a region if its flux across every closed surface in the region is zero.

Volume integral or Space integral. Let V be the volume bounded by the surface S. let $f(x, y, z)$ be a single valued function of position defined over V. Subdivide the volume V into n elements of volumes $\delta V_1, \delta V_2, \ldots, \delta V_n$. In each part δV_k, choose an arbitrary point $P_k (x_k, y_k, z_k)$. Form the sum $\sum_{k=1}^{n} f(P_k)\,\delta V_k$. Take the limit of the sum in such a ways that the largest of the volumes $\delta V_k \to 0$. This limit, if it exists, is called the volume integral of f over V and is denoted by $\iiint_V f\,dV$ or $\int_V f\,dV$.

If we divide the volume V into small cuboids by drawing lines parallel to the three co-ordinate axis, then

$$dV = dx\,dy\,dz \text{ and}$$

$$\therefore \quad \text{volume integral} = \iiint_V f\,dx\,dy\,dz\,.$$

Let $d\vec{S}$ be a vector (called vector area) of magnitude dS and direction that of $\hat{n}$. Then $d\vec{S} = \hat{n}\, dS$

$$\therefore \quad \iint_S \vec{F}\cdot\hat{n}\, dS = \iint_S \vec{F}\cdot d\vec{S}$$

Let α, β, γ be the angles which $\hat{n}$ makes with co-ordinate axes. If l, m, n are the direction-cosine of this outward normal, then $l = \cos\alpha, m = \cos\beta, n = \cos\gamma$.

$$\therefore \quad \hat{n} = (\cos\alpha)\hat{i} + (\cos\beta)\hat{j} + (\cos\gamma)\hat{k} = l\,\hat{i} + m\,\hat{j} + n\,\hat{k}$$

Let $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

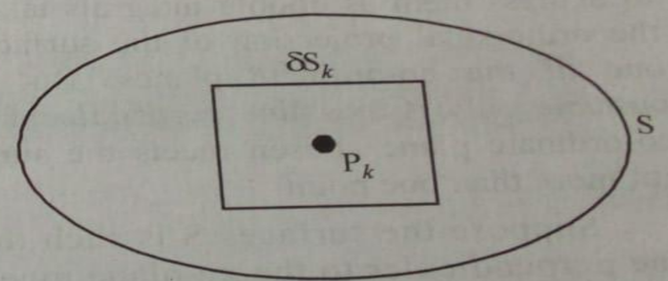
$$\therefore \quad \vec{F}\cdot\hat{n} = F_1\cos\alpha + F_2\cos\beta + F_3\cos\gamma = F_1\, l + F_2\, m + F_3\, n$$

SECTION-III

SURFACE AND VALUED INTEGRAL

Art-7. Surface Integral

Any integral which is to be evaluated over a surface is called a surface integral.



Let $f(x, y, z)$ be a single valued function defined over a surface S of finite area. Subdivide the area S into n elements of areas $\delta S_1, \delta S_2, \ldots, \delta S_n$. In each part δS, we choose an arbitrary point P_k $(x_k\ y_k,\ z_k)$. Form the sum $\sum_{k=1}^{n} f(P_k)\,\delta S_k$. Take the limit of this sum as $n \to \infty$ in such a way that the largest of the areas δS_k approaches zero. This limit if it exists, is called the surface integral of $f(x, y, z)$ over S and is denoted by

$$\iint_S f(x, y, z)\, dS \quad \text{or} \quad \int_S f\, dS.$$

Note 1. We know that

$$\text{Tangential line integral} = \int_A^B \vec{F} \cdot d\vec{r}\,.$$

In general, the value of this integral depends not only on the end points A and B of the path C but also on C. This line integral is said to be independent of path for every pair of end points A and B, the value of the integral is the same for all paths C starting from A and ending at B.

We have proved in above Art that tangential line integral of $\vec{F}$ is independent of path iff $\vec{F} = \nabla V$.

Note 2. From the result proved in above Art, we have :

Let $\vec{F}(x, y, z)$ be a vector point function defined and continuous in a region R of space. Then $\vec{F}$ is irrotational in R iff $\vec{F} = \nabla\phi$ where ϕ is a scalar point function.

Example 1. Find the work done when a force $\vec{F} = (x^2 - y^2 + x)\hat{i} - (2xy + y)\hat{j}$ moves a particle in xy-plane from (0, 0) to (1, 1) along the parabola $y^2 = x$.

Sol. Let C denote the arc of the parabola $y^2 = x$ from the point (0, 0) to the point (1, 1). The parametric equations of the parabola $y^2 = x$ can be taken as $x = t^2, y = t$. At the point (0, 0), $t = 0$ and at the point (1, 1), $t = 1$.

Now $\vec{F} = (x^2 - y^2 + x)\hat{i} - (2xy + y)\hat{j}$, $d\vec{r} = (dx)\hat{i} + (dy)\hat{j}$

$$\therefore \quad \text{work done} = \int_C \vec{F} \cdot d\vec{r} = \int_C [(x^2 - y^2 + x)\,dx - (2xy + y)\,dy]$$

$$= \int_0 \left[(x^2 - y^2 + x)\frac{dx}{dt} - (2xy + y)\frac{dy}{dt}\right] dt$$

$$= \int_0^1 [(t^4 - t^2 + t^2)(2t) - (2t^3 + t)(1)]\,dt$$

$$= \int_0^1 (2t^5 - 2t^3 - t)\,dt = \left[\frac{2t^6}{6} - \frac{2t^4}{4} - \frac{t^2}{2}\right]_0^1$$

$$= \left[\left(\frac{1}{3} - \frac{1}{2} - \frac{1}{2}\right) - (0 - 0 - 0)\right] = -\frac{2}{3}.$$

Example 14. Evaluate $\iiint_V \phi \, dV$, where $\phi = 45\, x^2 y$ and V is the closed region bounded by the planes $4x + 2y + z = 8, x = 0, y = 0, z = 0.$

Sol. $$\iiint_V \phi \, dV = \int_{x=0}^{2} \int_{y=0}^{4-2x} \int_{z=0}^{8-4x-2y} 45\, x^2 y \; dx\, dy\, dz$$

$$= 45 \int_{x=0}^{2} \int_{y=0}^{4-2x} [z]_0^{8-4x-2y}\, x^2 y \, dy\, dx$$

$$= 45 \int_{x=0}^{2} \int_{y=0}^{4-2x} x^2 y (8 - 4x - 2y)\, dy\, dx$$

$$= 45 \int_{x=0}^{2} \left[x^2 (8 - 4x) \frac{y^2}{2} - 2x^2 \frac{y^3}{3} \right]_0^{4-2x} dx$$

$$= 15 \int_0^2 x^2 (4 - 2x)^3 \, dx = 128 \qquad \text{(After simplification)}$$

Example 3. Evaluate $\int_0^1 \left\{ t\hat{i} + (t^2 - 2t)\hat{j} + (3t^2 + 3t^3)\hat{k} \right\} dt$

Sol. $\int_0^1 \left\{ t\hat{i} + (t^2 - 2t)\hat{j} + (3t^2 + 3t^3)\hat{k} \right\} dt$

$$= \hat{i}\int_0^1 t\,dt + \hat{j}\int_0^1 (t^2 - 2t)\,dt + \hat{k}\int_0^1 (3t^2 + 3t^3)\,dt$$

$$= \hat{i}\left[\frac{t^2}{2}\right]_0^1 + \hat{j}\left[\frac{t^3}{3} - t^2\right]_0^1 + \hat{k}\left[t^3 + \frac{3t^4}{4}\right]_0^1$$

$$= \hat{i}\left[\frac{1}{2} - 0\right] + \hat{j}\left[\left(\frac{1}{3} - 1\right) - (0 - 0)\right] + \hat{k}\left[\left(1 + \frac{3}{4}\right) - (0 + 0)\right]$$

$$= \frac{1}{2}\hat{i} - \frac{2}{3}\hat{j} + \frac{7}{4}\hat{k}$$