Relations

Definition: A **Relation** R from set A to set B is a subset of A × B.

- If $(a , b) \in R$, we say that "a is related to b", and write aRb.
- If $(a , b) \notin R$, we say that "a is not related to b", and write $a\not R b$.
- If $A = B$, we often say that $R \in A \times A$ is a relation on A.

Example: $A = (1, 2, 3)$ and $B = \{x, y, z\}$, and let

$$R = \{(1, y), (1, z), (3, y)\}.$$

Then R is a relation from A to B ? . **Yes--since *R* is a subset of *A* × *B***

With respect to this relation,

$$1Ry, 1Rz, 3Ry, \quad \text{but} \quad 1\not R x, 2\not R x, 2\not R y, 2\not R z, 3\not R x, 3\not R z$$

Relations

Example: Consider these relations on the set of integers:

$R_1 = \{(a,b) \mid a \le b\}$,
$R_2 = \{(a,b) \mid a > b\}$,
$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$,
$R_4 = \{(a,b) \mid a = b\}$,
$R_5 = \{(a,b) \mid a = b + 1\}$,
$R_6 = \{(a,b) \mid a + b \le 3\}$.

Which of these relations contain each of the pairs

(1,1), (1, 2), (2, 1), (1, −1), and (2, 2)?

Solution: Note that these relations are on an infinite set and each of these relations is an infinite set. Checking the conditions that define each relation, we see that

(1,1) is in R_1, R_3, R_4, and R_6:
(1,2) is in R_1 and R_6:
(2,1) is in R_2, R_5, and R_6:
(1, −1) is in R_2, R_3, and R_6 :
(2,2) is in R_1, R_3, and R_4.

Types of relations

Types of relations which are defined on a set *A*.

- Reflexive and Irreflexive Relations
- Symmetric and Antisymmetric Relations
- Transitive Relations

Definition: A relation R on a set A is **reflexive** if $(a,a) \in R$ for all $a \in A$.
Thus R is **not reflexive** if there exists $a \in A$ such that $(a, a) \notin R$.

Example: Consider the following relations on the set $A = \{1, 2, 3\}$:

$R_1 = \{(1, 1)(1, 2), (2, 1), (2, 2), (3, 3)\}$

$R_2 = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$

$R_3 = \Phi$

Determine which relation is reflexive.

Types of relations (Cont…)

Example: **The following relations on the integers are reflexive:**

$R_1 = \{(a,b) \mid a \leq b\}$,

$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$,

$R_4 = \{(a,b) \mid a = b\}$.

The following relations are not reflexive:

$R_2 = \{(a,b) \mid a > b\}$ (note that $3 \ngtr 3$),

$R_5 = \{(a,b) \mid a = b + 1\}$ (note that $3 \neq 3 + 1$),

$R_6 = \{(a,b) \mid a + b \leq 3\}$ (note that $4 + 4 \nleq 3$).

Types of relations (Cont...)

Definition: A relation R on a set A is **symmetric** if whenever aRb then bRa, i.e., if whenever $(a, b) \in R$ then $(b, a) \in R$.

Thus R is **not symmetric** if there exists $a, b \in A$ such that $(a, b) \in R$ but $(b, a) \notin R$.

Example: Consider the following relations on the set $A = \{1, 2, 3\}$:

$$R_1 = \{(1, 1)(1, 2), (2, 1), (2, 2), (3, 3)\}$$

$$R_2 = \{(1, 1), (1, 2), (2, 2)\}$$

Determine which relation is symmetric.

Types of relations (Cont...)

Definition: A relation R on a set A is **antisymmetric** if whenever aRb and bRa then a = b.

The contrapositive of this definition is that R is antisymmetric if whenever $a \neq b$, then either $(a,b) \notin R$ or $(b,a) \notin R$.

Definition: A relation R is **not antisymmetric** if there exist $a, b \in A$ such that $(a,b) \in R$ and $(b, a) \in R$ but $a \neq b$.

Example: Consider the following relations on the set A = {1, 2, 3}:

$$R_1 = \{(1, 1)(1, 2), (2, 1), (2, 2), (3, 3)\}$$

$$R_2 = \{(1, 1), (1, 2)\}$$

Determine which relation is antisymmetric.

Note: Not symmetric ≠ antisymmetric .

Types of relations (Cont…)

Definition: A relation R on a set A is **transitive** if whenever aRb and bRc then aRc, that is, if whenever $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$.

Thus R is **not transitive** if there exist $a, b, c \in R$ such that $(a,b) \in R$ and $(b, c) \in R$ but $(a,c) \notin R$.

If such a, b and c not exist, then R is transitive.

Example: Consider the following relations on the set $A = \{1, 2, 3\}$:

$R_1 = \{(1, 1), (1, 2), (2, 3), (1, 3)\}$

$R_2 = \{(1, 1), (1, 2), (2,2), (2,3)\}$

$R_3 = \{(1, 1), (1, 2), (1,3), (3,3)\}$

Determine which relation is transitive.

Equivalence relation

Definition: A relation R on a set A is called an **equivalence** relation if R is reflexive, symmetric, and transitive.

- It follows three properties:
  1) For every $a \in A$, aRa.
  2) If aRb then bRa.
  3) If aRb and bRc, then aRc.

Example: Consider the following relation on the set $A = \{1, 2, 3, 4\}$:

$$R = \{(1, 1), (1, 2), (2,1), (2,2), (3,4), (4,3), (3,3), (4, 4)\}$$

Determine whether this relation is equivalence or not.

The relation R is equivalence because R is reflexive, symmetric and transitive.

Relations (Cont…)

Definition: The **domain** of relation R is the set of all first elements of the ordered pairs which belong to R, denoted by Dom(R).

Definition: The **range** is the set of second elements of the ordered pairs which belong to R, denoted by Ran(R).

Example: $A = (1, 2, 3)$ and $B = \{x, y, z\}$, and consider the relation

$$R = \{(1, y), (1, z), (3, y)\}.$$

Find the domain and range of R.

The domain of R is Dom(R) $= \{1, 3\}$

The range of R is Ran(R) $= \{y, z\}$

Inverse Relations

Definition: Let R be any relation from set A to B. The inverse of R, denoted by R^{-1}, is the relation from B to A denoted by

$$R^{-1} = \{(b , a) \mid (a , b) \in R\}$$

Example: let $A = \{1, 2, 3\}$ and $B = \{x, y, z\}$. Find the inverse of

$$R = \{(1 , y), (1 , z), (3 , y)\}$$

Solution: $R^{-1} = \{(y , 1), (z , 1), (y , 3)\}$

- If R is any relation, then $(R^{-1})^{-1} = R$.
- The domain and range of R^{-1} are equal to the range and domain of R, respectively.
- If R is a relation on A, then R^{-1} is also a relation on A.